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Abstract: In this paper, algorithms to compute robust control invariant sets are proposed for linear continuous-time systems subject to additive but bounded disturbances. Robust control invariant sets of linear time invariant systems are achieved by logarithmic norm. Robust control invariant sets of linear uncertain systems, which are level sets of the storage functions, are obtained by solving functional differential inequality. Simulation shows that the proposed algorithms can yield improved minimal volume robust control invariant sets approximations in comparison with the schemes in the existing literature.

Keywords: Bounded disturbance, control invariant set, differential inequality, linear system, Logarithmic norm.

1. INTRODUCTION

Robust control invariant set refers to a bounded state space region in which the system state can be confined, despite the presence of disturbances or uncertainties, through the application of a control law [1–3]. Invariant set plays an important role in the robustness analysis and synthesis of controllers for uncertain systems [4–9]. Furthermore, terminal set, which is an invariant set of considered systems under a control law, is needed in the formulation of model predictive control [11, 12]. Both recursive feasibility and asymptotic stability can be assured by the appropriately chosen terminal set and terminal control law [13–16]. The existence of stabilizing control laws for discrete-time linear constrained system on controlled invariant sets is proved in [10].

The effect of disturbances or perturbations is a common issue in the analysis and synthesis of dynamical systems. In a typical situation, the value of the perturbations or disturbances is unknown but bounded. If the perturbations or disturbances are non-vanishing, i.e., do not disappear while time goes to infinite, asymptotic stability cannot be achieved in general. However, under certain conditions, ultimate boundedness or robust control invariance can be guaranteed. Recently, many research efforts have been devoted to computing robust invariant sets. The approximation of the minimal robust invariant set for an asymptotically stable discrete time linear system is considered in [17], which allows one to *a priori* specify the accuracy of the approximation. A procedure and theoretical results are presented for the problem of determining a minimal robust control invariant set for a linear discrete-time system subject to unknown and bounded disturbances, where the procedure computes via the solving of a linear programming [18]. A family of parameterized robust control invariant sets for linear discrete time systems subject to additive but bounded disturbances is characterized in [19], where the existence of a member of the introduced family of parameterized robust control invariant sets can be verified by solving a tractable convex optimization problem in the linear convex case. Nonlinear control law (piecewise affine in the most frequently encountered cases), rather than linear control law [17], is adopted in [20], where the existence of two families of robust control invariant sets is established. The problem of evaluating robust control invariant sets for linear discrete time systems subject to state and input constraints as well as additive disturbances is considered in [21], where a numerically efficient algorithm for the computation of full-complexity polytopic robust control invariant sets are presented. A method is presented in [22] for determining robust invariant sets and associated linear feedback laws for discrete-time linear systems with polytopic uncertainty. The problem of computing a maximal controlled invariant low-complexity polytopic set is then formulated as a bilinear constrained problem, and a relaxation of this problem is derived as an iterative sequence of convex programs. An algorithm is proposed in [2] to compute robust control invariant sets for linear discrete-time systems subject to norm-bounded model uncertainties, additive disturbances and polytopic

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constraints on the input and state. The proposed scheme explicitly takes account of norm-bounded model uncertainty and does not require any iterative computations or initial estimates of the invariant set. The construction of robust control invariant sets of systems with matched nonlinearity and a particular class of piecewise affine systems are exploited in [23]. The maximum controlled invariant set for discrete as well as continuous time nonlinear dynamical systems are characterized as the solution of a finite-dimensional linear programming problem [24, 25]. The problem of determining the maximal contractively invariant ellipsoids for discrete time systems with multiple inputs under saturated linear feedback is considered in [26]. An algebraic computational approach to determining such maximal contractively invariant ellipsoids are proposed. A real function is called a D. C. function if it is a difference of two convex functions. A method for computing a convex robust control invariant set for discrete-time nonlinear uncertain systems is presented in [27], which resorts to the properties of D. C. functions. The ellipsoidal invariant set of fractional order systems subject to actuator saturation is investigated in [28], where the lyapunov direct approach and fractional order inequality are applied to.

In this paper, algorithms for computing robust control invariant sets are proposed. Exploiting the properties of logarithmic norm, a sufficient condition for computing the robust control invariant set of linear time-invariant systems is provided. Based on a functional inequality, robust control invariant sets of linear systems with perturbations and disturbances are considered. The proposed methods are interesting for two reasons. On one hand, it indicates further that robust control invariant sets are useful tool for controller synthesis of linear uncertain systems, and guarantees robust stability for an adequate set of initial conditions. On the other hand, it provides a fairly simple algorithmic procedure.

This paper is organized as follows. Section 2 includes problem setup, and the definition of robust (control) invariant sets. Section 3 deals with the problem of calculating robust control invariant sets for linear time-invariant system by logarithmic norm. Section 4 reviews a general scheme to compute robust control invariant sets, and discusses robust control invariant sets of linear uncertain systems, namely polytopic and norm-bounded uncertain systems. Finally, Section 6 concludes the paper.

1.1. Notations and basic definitions

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{Z}$ the set of integer numbers. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the 2-norm. Suppose that $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of eigenvalues of the matrix M. Moreover, the symbol $\star$ is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b^T \\ b & c \end{bmatrix} = \begin{bmatrix} a & \star \\ b & c \end{bmatrix}$. The term $\mathrm{Co}\{\cdot\}$ denotes the convex hull of a set. For a symmetric matrix $X \in \mathbb{R}^{n\times n}$, let $X \succ 0 (X \succeq 0)$ denote that X is a positive (semi-) definite matrix, and $X \prec 0 (X \preceq 0)$ denote that X is a negative (semi-) definite matrix.

2. PROBLEM SETUP

Consider the following linear uncertain systems:

$$\begin{aligned}\dot{x}(t) =&(A+\Delta A)x(t)+(B+\Delta B)u(t)\\ &+(B_w+\Delta B_w)w(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state of the system, $u(t) \in \mathbb{R}^{n_u}$ the control input. The signal $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance or uncertainty, which is unknown but bounded, and lies in a compact set

$$\mathcal{W} := \{w \in \mathbb{R}^{n_w} \mid \|w\| \le w_{max}\},$$

i.e., $w(t) \in \mathcal{W}$ for all $t \ge 0$. The system matrices $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_u}$ and $B_w \in \mathbb{R}^{n_x\times n_w}$ are constant matrices, $\Delta A \in \mathbb{R}^{n_x\times n_x}$, $\Delta B \in \mathbb{R}^{n_x\times n_u}$ and $\Delta B_w \in \mathbb{R}^{n_x\times n_w}$ are compatible uncertain matrices.

Note that $F(x) := (A+\Delta A)x + (B+\Delta B)u + (B_w + \Delta B_w)w$ is a real-valued functional or a differential inclusion rather than a function [29]. Accordingly, the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t) \tag{2}$$

is the nominal system of the given system (1).

Next, the definition of robust control invariant set and a technical assumption are introduced. It shows that a robust (control) invariant set is a region where the trajectory generated by the dynamical systems (under control) remains confined in the region if the initial state lies in it.

Definition 1 [5]**:** A set $\Omega \subset \mathbb{R}^n$ is a robust control invariant set for the system (1) if there exists a feedback control law $\kappa(\cdot)$ such that for all $x(t_0) \in \Omega$, $x(t) \in \Omega$ for all $w \in \mathcal{W}$ and for all $t \ge t_0$.

Furthermore, if the control law $\kappa(\cdot)$ is determined *a priori*, Ω is a *robust invariant set* of the closed-loop system.

Assumption 1: The pair (A,B) is stabilizable.

Remark 1: For the nominal systems $\dot{x} = Ax(t) + Bu(t) + B_w w(t)$, there exists a linear control law Kx such that $A+BK$ is Huiwitz.

3. ROBUST CONTROL INVARIANT SETS BASED ON LOGARITHMIC NORM

In this section, the concept and characterization of logarithmic norm are introduced. After that the scheme to obtain robust control invariant sets based on logarithmic norm is discussed.

The logarithmic norm of a real matrix M is defined as [30, 31]

$$\mu(M) = \lim_{h\to 0^+} \frac{\|I+hM\|-1}{h}, \tag{3}$$

where the symbol $\|\cdot\|$ represents any matrix norm defined in the inner product space with an inner product $\langle x,y\rangle$, and I is the dimensional compatible identity matrix. Note that the name of logarithmic norm is original from estimating the logarithm of the norm of solutions to the differential equation $\dot{x} = Mx$, i.e.,

$$\frac{d}{dt^+}\log\|x\| \le \mu(M) \quad \text{or} \quad \frac{d}{dt^+}\|x\| \le \mu(M)\|x\|,$$

where d/dt^+ is the upper right Dini derivative. That is, the maximal growth rate of $\log\|x\|$ is $\mu(M)$.

For the standard inner product $\langle x,y\rangle := x^T y$, the logarithmic norm of the matrix M is given by [32]

$$\mu_2(M) = \sup_{\|x\|=1} \langle x, Mx\rangle, \tag{4}$$

which is equivalent to

$$\mu_2(M) = \lambda_{max}\left(\frac{M+M^T}{2}\right).$$

Remark 2: If the standard inner product $\langle x,y\rangle = x^T y$ is adopted, there is *no* guarantee that there exists a matrix $K \in \mathbb{R}^{n_u}\times\mathbb{R}^{n_x}$ such that $\mu_2(A+BK) \le 0$ for any pair (A,B) even if (A,B) is stabilizable. Thus, in the following the inner product $\langle x,y\rangle := x^T Hy$ is introduced, where H is a positive definite matrix.

Suppose that x, y are finitely many dimensions, and $\langle x,y\rangle := x^T Hy$, where H is a positive definite matrix, then

$$\mu_H(M) = \max\left\{\lambda \mid \det(M^T H + HM - 2\lambda H) = 0\right\}, \tag{5}$$

where $\det(\cdot)$ is the determinant of a given matrix [32]. Eq. (5) specifies $\mu_H(M)$ as a solution to a generalized eigenvalue problem. If $H = I$, $\mu_H(M)$ becomes $\mu_2(M)$.

For the sake of computation purposes, Eq. (5) can be reformulated as the form of the following matrix inequality:

$$\mu_H(M) = \min\left\{\beta \mid M^T H + HM - 2\beta H \preceq 0\right\}. \tag{6}$$

Next lemma collects a subset of well-known results, whose proof can be found in [31, 32], or reference therein.

Lemma 1: Let M and N be square matrices, and $\lambda > 0$. Then

- $\mu(\lambda M) = \lambda\mu(M)$,
- $\mu(M+N) \le \mu(M)+\mu(N)$,
- $\|e^{Mt}\| \le e^{\mu(M)t}$.

It concludes immediately from Lemma 1 that $\|e^{Mt}\| \le 1$ for all nonnegative t if and only if $\mu(M) \le 0$ [31, 32]. Eq. (4) together with Eq. (5) show that logarithmic norm is *not* a real norm since it can be negative in some cases.

Since (A,B) is stabilizable, there exist a state feedback matrix $K \in \mathbb{R}^{n_u\times n_x}$ and positive matrix $P \in \mathbb{R}^{n_x\times n_x}$ such that $(A+BK)^T P + P(A+BK) \preceq 0$ [33]. Compared with Eq. (6), $\mu_p(A+BK) \le 0$, i.e., $\|e^{(A+BK)t}\|_p \le 1$ while $(A+BK)^T P + P(A+BK) \preceq 0$.

For simplicity, denote $A_{cl} := A+BK$. The evolution of system (2) under the control law $u := Kx$ can be written as

$$x(t) = e^{A_{cl}t}x(0) + \int_0^t e^{A_{cl}(t-\tau)}B_w w(\tau)d\tau, \tag{7}$$

where $x(0)$ is the initial state of the system (2).

The exogenous disturbance $w(t)$ is bounded in the inner space $\langle x,x\rangle := x^T Px$, that is $\|w\|_p \le w_{p,\max}$ in terms of the equivalent induced matrix norm, where $w_{p,\max}$ is a given scalar. For example, if $\langle x,y\rangle = x^T Py$, then $w_{p,\max} \le \sqrt{\lambda_{\max}(P)}w_{max}$.

The following theorem provides a way to construct robust control invariant sets of system (2).

Theorem 1: Consider system (2). Suppose that there exist $K \in \mathbb{R}^{n_u\times n_x}$ and a positive definite matrix $P \in \mathbb{R}^{n_x\times n_x}$ such that $\mu_p(A_{cl}) < 0$. Then, the set

$$\Omega_0 := \left\{x \in \mathbb{R}^{n_x} \mid \|x\|_p \le \frac{\|B_w\|_p w_{p,\max}}{-\mu_p(A_{cl})}\right\} \tag{8}$$

is a robust invariant set of the system (2).

Proof: The inequality $\|e^{A_{cl}t}\|_p \le e^{\mu_p(A_{cl}t)}$ is used to estimate the solution (7):

$$\begin{aligned}
\|x(t)\|_p \le& \left\|e^{A_{cl}t}x(0)\right\|_p + \int_0^t \left\|e^{A_{cl}(t-\tau)}B_w w(\tau)\right\|_p d\tau \\
\le& e^{\mu_p(A_{cl}t)}\|x(0)\|_p \\
&+ \left(1-e^{\mu_p(A_{cl}t)}\right)\frac{\|B_w\|_p w_{p,\max}}{-\mu_p(A_{cl})} \\
=& e^{\mu_p(A_{cl}t)}\left(\|x(0)\|_p - \frac{\|B_w\|_p w_{p,\max}}{-\mu_p(A_{cl})}\right) \\
&+ \frac{\|B_w\|_p w_{p,\max}}{-\mu_p(A_{cl})}.
\end{aligned}$$

Thus, if $x(0) \in \Omega_0$, then $x(t) \in \Omega_0$ for all $t \ge 0$. Therefore, Ω_0 is an invariant set of system (2). □

Remark 3: If there exists a matrix $K \in \mathbb{R}^{n_u}\times\mathbb{R}^{n_x}$ such that $\mu_2(A+BK) \le 0$ for the pair (A,B), then the closed-loop system has the “optimal" transient behavior since $\left\|e^{A_{cl}t}\right\| \le Me^{\mu_2(A+BK)t}$ with $M = 1$.

The above discussion also gives a valuable insight into part b) of Problem 1, in the chapter of “Problem 6.3" [34], for the system which has the property of $\mu_2(A+BK) \le 0$.

Theorem 1 shows that there exist control invariant sets, if the corresponding nominal systems are exponentially stable. In the next section, robust control invariant sets of linear systems with uncertainties are considered.

4. ROBUST CONTROL INVARIANT SETS BASED ON DIFFERENTIAL INEQUALITY

Definition 2 [33]**:** A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Lemma 2 [35]**:** Let $S : \mathbb{R}^{n_x} \to [0,\infty)$ be a continuously differentiable function and $\alpha_1(\|v\|) \le S(v) \le \alpha_2(\|v\|)$. Suppose there exist $\alpha > 0$ and $\mu > 0$ such that

$$\frac{d}{dt}S(v) + \alpha S(v) - \mu w^T w \le 0, \quad \forall w \in \mathcal{W}, \tag{10}$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions. Then, for a general nonlinear systems $\dot{v} = f(v,w)$, the system trajectory will remain in the set Ω if $v(t_0) \in \Omega$, where $f(\cdot,\cdot)$ is a Lipschitz function on v and w, and

$$\Omega := \left\{ v \in \mathbb{R}^{n_x} \mid S(v) \le \frac{\mu w_{max}^2}{\alpha} \right\}. \tag{11}$$

The following proof gives an interpretation of Lemma 2 from the optimization theory perspective.

Proof: For nonlinear systems $\dot{v} = f(v,w)$, $v \notin \Omega$ is equivalent to $S(v) > \frac{\mu w_{\max}}{\alpha}$, and $w \in \mathcal{W}$ is equivalent to $w(t)^T w(t) \le w_{\max}^2$ for all $t \ge 0$. In accordance with S-procedure [29], it is sufficient for $\dot{S}(v) \le 0$ for all $x \notin \Omega$ and for all $w \in \mathcal{W}$, if it holds that

$$-\frac{dS(v)}{dt} - \alpha\left(S(v) - \frac{\mu w_{\max}^2}{\alpha}\right) - \mu\left(w_{\max}^2 - w(t)^T w(t)\right) \ge 0,$$

with $\mu > 0$ and $\alpha > 0$. That is, $\frac{d}{dt}S(v) + \alpha S(v) - \mu w^T w \le 0$ for all $w(t) \in \mathcal{W}$. □

4.1. Robust control invariant sets of linear uncertain systems

Our goal in this subsection is to find a robust control invariant set for the given differential inclusion (1).

4.1.1 Polytopic model of linear uncertain systems

If $[A + \Delta A, B + \Delta B, B_w + \Delta B_w] \in \Sigma$, where

$$\Sigma := \mathrm{Co}\left\{ \begin{bmatrix} A_1 & B_1 & B_{w,1} \end{bmatrix}, \ldots, \begin{bmatrix} A_N & B_N & B_{w,N} \end{bmatrix} \right\}, \tag{12}$$

then, Σ is a polytopic differential inclusion of system (1), $\begin{bmatrix} A_i & B_i & B_{w,i} \end{bmatrix}$, $i \in \mathbb{Z}_{[1,N]}$, is the vertex matrix of the set Σ, and N is the number of the vertex matrix.

Theorem 2: Suppose that there exist a positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, a possible non-square matrix $Y \in \mathbb{R}^{n_u \times n_x}$, and scalars $\alpha > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} (A_iX + B_iY)^T + A_iX + B_iY + \alpha X & B_{w,i} \\ \star & -\mu I \end{bmatrix} \preceq 0, \tag{13}$$

for all $i \in \mathbb{Z}_{[1,N]}$. Then, with $u(t) := Kx(t)$ and $S(x(t)) := x(t)^T P x(t)$, inequality (10) is satisfied for the polytopic linear differential inclusion (12), where $P := X^{-1}$ and $K := YX^{-1}$. Therefore, the system (1) is robustly invariant in the set

$$\Omega := \left\{ x \in \mathbb{R}^{n_x} \mid x^T P x \le \frac{\mu w_{max}^2}{\alpha} \right\}.$$

Proof: Pre-and post-multiplying (13) by $\mathrm{diag}(P, I)$ yields

$$\begin{bmatrix} (A_i + B_iK)^T P + P(A_i + B_iK) + \alpha P & PB_{w,i} \\ \star & -\mu I \end{bmatrix} \preceq 0 \tag{14}$$

for all $i \in \mathbb{Z}_{[1,N]}$. Multiplying (14) from both sides with $\begin{bmatrix} v(t)^T & w(t)^T \end{bmatrix}$ and $\begin{bmatrix} v(t)^T & w(t)^T \end{bmatrix}^T$, respectively, due to Eq.(1), it follows that the inequality

$$\frac{d}{dt}\left(v(t)^T P v(t)\right) + \alpha v(t)^T P v(t) - \mu w(t)^T w(t) \le 0$$

is satisfied for all $w(t) \in \mathcal{W}$. Therefore, inequality (10) holds for the system (1). □

Remark 4: Compare (13) with the condition (6.28) in [29] and (4.23) in [8], an extra freedom, parameter μ, is introduced which will reduce the conservativeness of the involved optimization problem.

4.1.2 Norm-bounded model of linear uncertain systems

If $[A + \Delta A, B + \Delta B, B_w + \Delta B_w] \in \Sigma$, where

$$\Sigma := \{(\mathcal{A}, \mathcal{B}, \mathcal{B}_w) \mid \mathcal{A} = A + M\Delta(t)N_1, \quad \mathcal{B} = B + M\Delta(t)N_2, \mathcal{B}_w = B_w + M\Delta(t)N_3\},$$

M, N_1, N_2 and N_3 are known matrices with the appropriate dimensions and $\Delta(t)$ is a time-varying norm-bounded matrix satisfying

$$\bar{\sigma}(\Delta(t)) \le 1.$$

Then, Σ is a norm-bounded differential inclusion of system (1).

In the proof of the next theorem, the following lemma is needed.

Lemma 3 [36]**:** Given matrices Y, H, E of appropriate dimensions and with Y being symmetrical, then

$$Y + HFE + E^T F^T H^T \preceq 0$$

for all F satisfying $F^T F \le I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$Y + \varepsilon HH^T + \varepsilon^{-1} E^T E \preceq 0.$$

Parallel to Theorem 2, we have the following theorem:

Theorem 3: Suppose that there exist a positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, a possibly non-square matrix $Y \in \mathbb{R}^{n_u \times n_x}$, and scalars $\alpha > 0$, $\mu > 0$ and $\varepsilon > 0$ such that

$$\begin{bmatrix} \Gamma & B_w & (N_1X+N_2Y)^T \\ \star & -\mu I & N_3^T \\ \star & \star & -\varepsilon I \end{bmatrix} \preceq 0, \tag{15}$$

where $\Gamma := (AX+BY)^T + AX + BY + \alpha X + \varepsilon MM^T$. Then, with $u := Kx$ and $S(x) := x^T Px$, inequality (10) is satisfied for the system (1), where $P := X^{-1}$ and $K := YX^{-1}$. Therefore, the system (1) is robustly invariant in the set $\Omega := \left\{ x \in \mathbb{R}^{n_x} \mid x^T Px \leq \frac{\mu w_{max}^2}{\alpha} \right\}$.

Proof: Performing a congruence transformation on (15) with the nonsingular matrix diag$\{P, I, I\}$, the following inequality is obtained

$$\begin{bmatrix} P\Gamma P & PB_w & (N_1+N_2K)^T \\ \star & -\mu I & N_3^T \\ \star & \star & -\varepsilon I \end{bmatrix} \preceq 0. \tag{16}$$

By the Schur complement, (16) is equivalent to

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) + \alpha P & PB_w \\ \star & -\mu I \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} (N_1+N_2K)^T \\ N_3^T \end{bmatrix} \begin{bmatrix} N_1+N_2K & N_3 \end{bmatrix} + \varepsilon \begin{bmatrix} PM \\ 0 \end{bmatrix} \begin{bmatrix} M^T P & 0 \end{bmatrix} \preceq 0.$$

Since $\bar{\sigma}(\triangle(t)) \leq 1$ and $\varepsilon > 0$, due to Lemma 3, the forgoing equation is equivalent to

$$\begin{bmatrix} \prod & PB_w \\ \star & -\mu I \end{bmatrix} \preceq 0, \tag{17}$$

where $\prod := A_s^T P + PA_s + \alpha P$ with $A_s := A + M\triangle N_1 + (B + M\triangle N_2)K$. Multiplying (17) from both sides with $\begin{bmatrix} v(t)^T & w(t)^T \end{bmatrix}$ and $\begin{bmatrix} v(t)^T & w(t)^T \end{bmatrix}^T$, respectively, inequality (10) holds for the system (1). □

4.2. Optimization of robust control invariant sets

In order to limit the effects of disturbances or perturbations, the minimal robust control invariant set is recommended. The volume of ellipsoid centered at the origin Ω is proportional to det$\left(\frac{\mu w_{\max}^2}{\alpha} X\right)$ [29], which is not convex, but monotonic transformations can render it convex. The geometric mean of the eigenvalues [37], which leads to minimization of det$(\alpha X)^{\frac{1}{n_x}}$, can be used to solve the determinant maximization problem by YALMIP : a toolbox for modeling and optimization in Matlab.

The minimization problem of the ellipsoid Ω can be formulated as

Problem 1:

$$\underset{X,Y,\alpha,\mu}{\text{maximize}} \ \det\left(\frac{\mu w_{\max}^2 X}{\alpha}\right)^{\frac{1}{n_x}} \tag{18}$$

subject to (13)

or

Problem 2:

$$\underset{X,Y,\alpha,\mu,\varepsilon}{\text{maximize}} \ \det\left(\frac{\mu w_{\max}^2 X}{\alpha}\right)^{\frac{1}{n_x}} \tag{19}$$

subject to (15)

Remark 5: Both Problem 1 and Problem 2 are not linear matrix inequalities (LMIs) optimization problems since there exist terms of αX and $\frac{\mu w_{\max}^2 X}{\alpha}$. In order to find a possible minimum robust control invariant set by LMI Toolbox, a search over α and μ, or α, μ and ε, is required, respectively.

Remark 6: Both state and input constraints can be considered in the framework of multi-objective optimization solved using LMI [4, 38].

5. ILLUSTRATIVE EXAMPLES

5.1. Example 1

Consider an open-loop unstable system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), \tag{20}$$

with $A = \begin{bmatrix} -1.1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The disturbance $w \in \mathcal{W} \subset \mathbb{R}^1$, where

$$\mathcal{W} := \left\{ w \in \mathbb{R}^1 \mid -0.1 \leq w \leq 0.1 \right\}. \tag{21}$$

Choosing $P = I$ in Theorem 1, the logarithmic norm of system (20) with a linear control law $u = Kx$ can be obtained by solving the following optimization problem

$$\underset{K,\lambda}{\text{minimize}} \ \lambda$$

subject to

$$2\lambda - (A+BK) - (A+BK)^T \succeq 0.$$

The obtained logarithmic norm is $\lambda = -0.5562$. Thus, $\|e^{(A+BK)t}\|_2 \leq e^{-0.5562t}$. With the control gain $K = \begin{bmatrix} 0.2836 & 2.2992 \end{bmatrix}$, the robust control invariant set is

$$\Omega_0 = \left\{ x \in \mathbb{R}^2 \mid x^T x \leq 0.1798 \right\}.$$

The control invariant set yielded of the Theorem 1 is shown by the dashed ellipsoid in Fig. 1. The control invariant set given by Theorem 2 is represented by the solid ellipsoid, the control invariant set given by Eq.(6.28) in

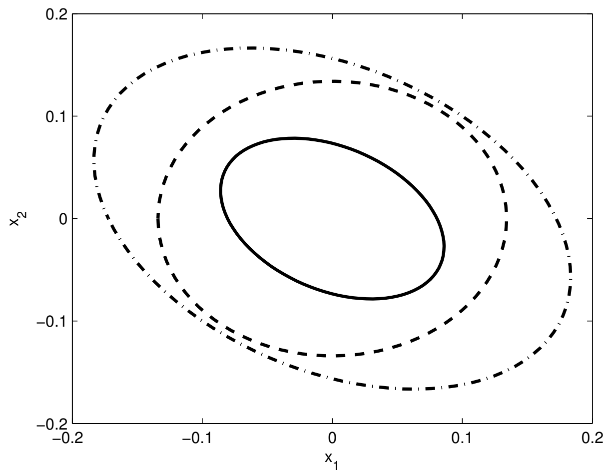

Fig. 1. Control invariant sets of the system (20).

[29] is represented by the dashed-dotted ellipsoid, respectively, where $\alpha = 4$ is chosen. Since the aim to design a robust control law is to limit the effect of disturbances or perturbations on system dynamics, it can be seen from Fig. 1 that Theorem 2 is the least conservative among the three schemes.

5.2. Example 2

Consider an uncertain linear system

$$\begin{aligned} \dot{x}_1 &= 2(1-\lambda)x_1 + x_2 + \lambda u, \\ \dot{x}_2 &= x_1 - 8(1-\lambda)x_2 + \lambda u + w \end{aligned} \tag{22}$$

with $\lambda \in [0.2, 0.8]$. The disturbance $w \in \mathcal{W} \subset \mathbb{R}^1$, c.f. (21).

The vertex matrices of the polytopic model (12) are $A_1 = \begin{bmatrix} 1.6 & 1 \\ 1 & -6.4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.4 & 1 \\ 1 & -1.6 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^T$, $B_2 = \begin{bmatrix} 0.8 & 0.8 \end{bmatrix}^T$, $B_{w1} = B_{w2} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

Solving Problem 1 iteratively to get a robust control invariant set of the system (22)

$$\Omega_0 = \left\{ x \in \mathbb{R}^{n_x} \mid x^T P x \le \alpha \right\}$$

with $P = \begin{bmatrix} 0.6789 & 0.0951 \\ 0.0951 & 0.2207 \end{bmatrix}$ and $\alpha = 0.0368$. The corresponding linear control law is $u = \begin{bmatrix} -16.0101 & -3.1349 \end{bmatrix} x$.

6. CONCLUSIONS

In this paper, schemes for the computation of robust control invariant sets were proposed for systems with norm-bounded uncertainties or polytopic uncertainties, and with additive but bounded disturbances. Logarithmic norm as well as the functional inequality was used to design robust control invariant sets. Furthermore, robust control invariant sets and the corresponding state feedback control law can be solved through an LMI optimization problem.

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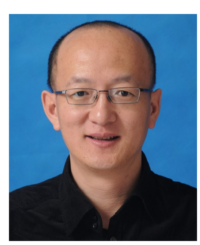

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